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HW 3

Math 583B
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1. a.) Show asymptotic behavior of $I(\tau)$ where

$$I(\tau) = \int_a^b f(k)e^{i\tau S(k)}\,dk \tag{1}$$

Where $-\infty < a < b < \infty$

Solution. Start with definition of $S'(a) = 0$, $S(a) \neq 0$ and $S''(a) \neq 0$

$$S(k) \sim S(a) + \frac{1}{2}S''(a)(k-a)^2 \tag{2}$$

Apply approximation to (1)

$$\begin{aligned} I(\tau) &\sim f(a)\int_a^b e^{i\tau S(a)}e^{i\tau\frac{1}{2}S''(a)(k-a)^2}\,dk && (3)\\ &\sim f(a)e^{i\tau S(a)}\int_a^b e^{i\tau\frac{1}{2}S''(a)(k-a)^2}\,dk && (4)\end{aligned}$$

Make the substitution $u = k - a$

$$I(\tau) \sim f(a)e^{i\tau S(a)}\int_0^{b-a} e^{i\tau\frac{1}{2}S''(a)u^2}\,du \tag{5}$$

Make substitution $v = \sqrt{\frac{\tau}{2}}u$ (keep in mind $\tau \to \infty$)

$$I(\tau) \sim \sqrt{\frac{2}{\tau}}f(a)e^{i\tau S(a)}\int_0^{\infty} e^{iS''(a)v^2}\,dv \tag{6}$$

Now, just need to account for $sgn(S''(a))$ and make substitution $v' = \sqrt{S''(a)}v$

$$\begin{aligned} I(\tau) &\sim \sqrt{\frac{2\pi}{\tau S''(a)}}f(a)e^{i\tau S(a)}\int_0^{\infty} e^{isgn(S''(a))v'^2}\,dv' && (7)\\ &= \frac{1}{2}\sqrt{\frac{2\pi}{\tau S''(a)}}f(a)e^{i\tau S(a)}e^{i\frac{\pi}{4}sgn(S''(a))} && (8)\end{aligned}$$

b.) Find leading asymptotic behavior of

$$J_0(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(x\sin t)\,dt \tag{9}$$

Solution. Rewrite $J_0(x)$ in the form from a.) using the even property of the integrand and $f(k) = 1$

$$J_0(x) = \frac{1}{2}\frac{2}{\pi}\Re[\int_{\frac{-\pi}{2}}^{0} e^{ix\sin t}\,dt] \tag{10}$$

Define $S(t) = \sin t$, then $S'(t) = \cos t$ and $S''(t) = -\sin t$. Apply $a = -\frac{\pi}{2}$, so $S(a) = -1$ and $S''(a) = 1$. Now, we can use (8)

$$J_0(x) \sim \Re[\frac{1}{\pi}\sqrt{\frac{2\pi}{t}}e^{-ix}e^{i\frac{\pi}{4}}] \tag{11}$$

$$= \sqrt{\frac{2}{x\pi}}\cos(\frac{\pi}{4} - x) \tag{12}$$

Therefore, leading asymptotic behavior as $x \to \infty$ is $\sim \frac{1}{\sqrt{x}}$

2. Use data to determine a.) wavelength and b.) distance traveled of ocean waves

Solution. a.) Use given relations

$$k = \frac{gt^2}{4R^2} \tag{13}$$

$$\frac{R}{t} = \frac{gP_1}{4\pi} \tag{14}$$

Make combination

$$\frac{2\pi}{\lambda} = k = \frac{g}{4(\frac{R}{t})^2} = \frac{4\pi^2}{gP_1^2} \tag{15}$$

Solve for λ

$$\lambda = \frac{gP_1^2}{2\pi} \tag{16}$$

For the values given, the table is

Period(sec)	Wavelength(m)
21.0	687.8
18.8	551.3
16.7	435.0
15.2	360.4
13.9	301.4

b.) To calculate the distance traveled, I used the relation

$$d = vt = (\frac{R}{t})t = \frac{gP_1}{4\pi}t \tag{17}$$

Using a broad estimate the source of the waves can be determined from the following plot

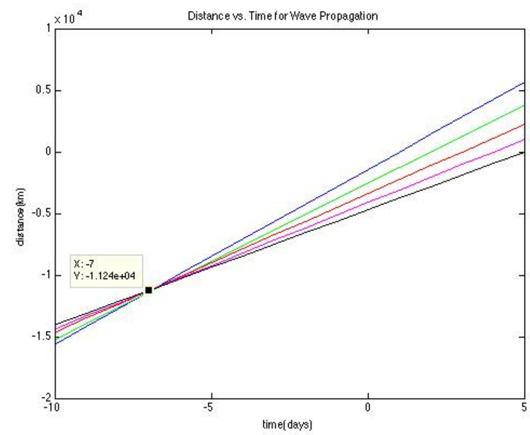

Figure 1: Distance is very roughly 11240 km

3. Find general distribution solutions to

$$u'(x) = \delta(x) \tag{18}$$

Solution. Fourier transform both sides to get

$$\mathcal{F}[u'(x)] = \mathcal{F}[\delta(x)] \tag{19}$$
$$(-ik)\mathcal{F}[u(x)] = 1 \tag{20}$$

Rearrange and inverse transform to get

$$\mathcal{F}^{-1}[\mathcal{F}[u(k)]] = i\mathcal{F}^{-1}[\frac{1}{k}] \tag{21}$$
$$= iPV[\frac{1}{k}] \tag{22}$$

Apply value for principal value integral

$$u(x) = \begin{cases} \frac{i}{2\pi}i\pi & \text{if } x < 0, \\ \frac{i}{2\pi}(-i\pi) & \text{if } x > 0, \end{cases} \tag{23}$$
$$= \begin{cases} -\frac{1}{2} & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x > 0, \end{cases} \tag{24}$$

Thus, the final solution is the Heaviside function shifted down by $\frac{1}{2}$.